Solution to Assignment 7

Supplementary Problems

1. Find a parametric curve $\gamma(t),\ t \in [0,1]$, which describes the triangle with vertices at $(0,0),(2,0)$ and $(2,5)$ in anticlockwise direction.

   Solution. The three sides of the triangle are given by

   $$\gamma_1(t) = (t,0),\ t \in [0,2];$$

   $$\gamma_2(t) = (2,t),\ t \in [0,5]$$

   and

   $$\gamma_3(t) = (1-t)(2,5)\ , t \in [0,1]\ .$$

   Now, we rescale γ_1 so that it is on $[0,1/3]$ by $\mathbf{c}_1(t) = (6t,0)$. Recale γ_2 so that it is on $[1/3,2/3]$ by $\mathbf{c}_2(t) = (2,15t-5),\ t \in [1/3,2/3]$ and γ_3 to be on $[2/3,1]$, that is, $\mathbf{c}_3(t) = 3(1-t)(2,5),\ t \in [2/3,1]$. Then $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$ is our desired curve. Note the solution is not unique.

2. Find the arc-length parametrization of the line segment $y = mx + b,\ x \in [0,2]$.

   Solution. Let the line segment be $C(t) = (t, mt+b),\ t \in [0,2]$. Then $s = \psi(t) = \int_0^t |C'(t)|\,dt = \sqrt{1+m^2}\ t$. Therefore, $t = \varphi(s) = \dfrac{s}{\sqrt{1+m^2}}$. The arc-length parametric curve is

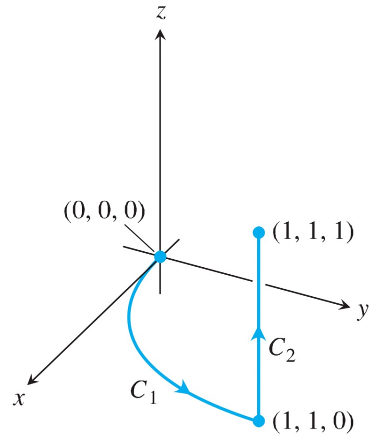
   $$\tilde{C}(s) = (s, ms + b\sqrt{1+m^2})/\sqrt{1+m^2},\ s \in [0, 2\sqrt{1+m^2}]\ .$$

Evaluating Line Integrals over Space Curves

15. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from (0, 0 to (1, 1, 1) (see accompanying figure) given by

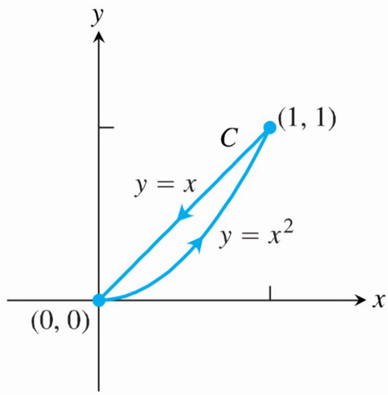
$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$$

$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$$



Line Integrals over Plane Curves

25. Evaluate $\int_C \left(x + \sqrt{y}\right) ds$ where C is given in the accompanying figure.



In Exercises 27–30, integrate f over the given curve.

30. $f(x, y) = x^2 - y, \quad C: \quad x^2 + y^2 = 4$ in the first quadrant from $(0, 2)$ to $(\sqrt{2}, \sqrt{2})$

Q15 $C_1: \vec{r}(t) = t\vec{i} + t^2\vec{j}$; $\vec{r}'(t) = \vec{i} + 2t\vec{j}$; $|\vec{r}'(t)| = \sqrt{1+4t^2}$.

$C_2: \vec{r}(t) = \vec{i} + \vec{j} + t\vec{k}$; $\vec{r}'(t) = \vec{k}$; $|\vec{r}'(t)| = 1$.

$$\therefore \int_{C_1 \cup C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

$$= \int_0^1 (t+t)\cdot\sqrt{1+4t^2}\,dt + \int_0^1 (1+1-t^2)\cdot 1\,dt$$

$$= \left[\frac{1}{6}(1+4t^2)^{\frac{3}{2}}\right]_0^1 + \left[2t - \frac{t^3}{3}\right]_0^1$$

$$= \frac{1}{6}(5\sqrt{5}-1) + \frac{5}{3} = \frac{1}{6}(5\sqrt{5}+9) //$$

Q25 Write $C = C_1 \cup C_2$, where

- $C_1: \vec{r}(t) = t\vec{i} + t^2\vec{j}$, $0 \le t \le 1$; $\vec{r}'(t) = \vec{i} + 2t\vec{j}$; $|\vec{r}'(t)| = \sqrt{1+4t^2}$.
- $C_2: \vec{r}(t) = (1-t)\vec{i} + (1-t)\vec{j}$, $0 \le t \le 1$; $\vec{r}'(t) = -\vec{i} - \vec{j}$; $|\vec{r}'(t)| = \sqrt{2}$

$$\therefore \int_C (x+\sqrt{y})\,ds = \int_{C_1 \cup C_2} (x+\sqrt{y})\,ds = \int_{C_1} (x+\sqrt{y})\,ds + \int_{C_2} (x+\sqrt{y})\,ds$$

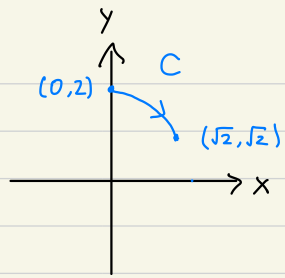
$$= \int_0^1 (t+t)\sqrt{1+4t^2}\,dt + \int_0^1 \left((1-t)+\sqrt{1-t}\right)\sqrt{2}\,dt$$

$$= \left[\frac{1}{6}(1+4t^2)^{\frac{3}{2}}\right]_0^1 + \sqrt{2}\left[t - \frac{t^2}{2} - \frac{2}{3}(1-t)^{3/2}\right]_0^1$$

$$= \frac{1}{6}(5\sqrt{5}-1) + \sqrt{2}\left[(1-\tfrac{1}{2}) - \tfrac{2}{3}(-1)\right]$$

$$= \frac{1}{6}(5\sqrt{5}-1+7\sqrt{2}) //$$

Q30 Note that C can be parametrized as follows:



$C: \vec{r}(t) = 2\sin t\,\vec{i} + 2\cos t\,\vec{j},\ 0 \leq t \leq \frac{\pi}{4};$

$\vec{r}\,'(t) = 2\cos t\,\vec{i} - 2\sin t\,\vec{j}\ ;\ |\vec{r}\,'(t)| = 2.$

$$\therefore \int_C f\,ds = \int_0^{\frac{\pi}{4}} (4\sin^2 t - 2\cos t)\,2\,dt$$

$$= 4\int_0^{\frac{\pi}{4}} (1 - \cos 2t - \cos t)\,dt$$

$$= 4\left[t - \frac{\sin 2t}{2} - \sin t\right]_0^{\frac{\pi}{4}}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\sqrt{2}}{2}\right) = \pi - 2\sqrt{2} - 2\ //$$